Interval-valued logics

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In [5], the authors introduced Interval-Valued Monoidal Logic[1] (IVML). Its language is the language of Höhle's Monoidal Logic (ML,[3]) enriched with two unary connectives $\Box$ and $\Diamond$, and a constant $\overline{u}$. Its axioms are those of ML plus 15 new ones describing the behaviour of $\Box$, $\Diamond$ and $\overline{u}$. The deduction rules are modus ponens (MP, from ϕ and $\phi \rightarrow \psi$ infer ψ), generalization (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \rightarrow \psi$ infer $\Diamond\phi \rightarrow \Diamond\psi$).
In some way, ML can be seen as a special case of IVML. Indeed, it can be proven [7] that for all sets $T \cup \{\phi\}$ of ML-formulae, $T \vdash_{ML} \phi$ iff $\{\chi' | \chi \in T\} \vdash_{IVML} \phi'$ (where ψ' is the IVML-formula obtained by substituting $\Box p$ in ψ for every proposition variable p in ψ).
IVML is sound and complete with respect to the variety of triangle algebras. These are algebraic structures that describe interval-valued residuated lattices (IVRLs): (closed) interval-valued bounded lattices endowed with a product and implication that satisfy the residuation principle, such that the sublattice of exact intervals (i.e., intervals consisting of one element) is closed under product and implication. Table 1 shows to which mappings and interval in an IVRL the connectives and constant in IVML correspond. Also the notations in triangle algebras are included, in the second column. The soundness and completeness

Table 1. Semantic meaning of $\Box$, $\Diamond$ and $\overline{u}$.

IVML	triangle algebra	IVRL
$\Box$	ν	$p_v : [x, y] \mapsto [x, x]$
$\Diamond$	μ	$p_h : [x, y] \mapsto [y, y]$
$\overline{u}$	u	$[0, 1]$

of IVML w.r.t. triangle algebras and the connection between triangle algebras and IVRLs explains why this logic was called interval-valued.
IVML (and its extensions) enjoys the following deduction theorem:
$T \cup \{\phi\} \vdash_{IVML} \psi$ iff there is an integer n such that $T \vdash_{IVML} (\Box\phi)^n \rightarrow \psi$.

Numerous extensions of IVML can be defined. One of them is Interval-Valued Monoidal T-norm based Logic (IVMTL), which compares to IVML in more or

[1] IVML was called Triangle Logic in [5], but was recently renamed [7].

less the same way as MTL [2] compares to ML. IVMTL (introduced in [6] under the name Pseudo-linear Triangle Logic) is IVML extended with the axiom scheme $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$. The semantics of this logic are pseudo-prelinear triangle algebras, i.e., triangle algebras in which the subalgebra of exact elements is an MTL-algebra. IVMTL and its extensions are even pseudo-chain complete [6], which means that we can restrict its semantics to pseudo-linear triangle algebras, i.e., triangle algebras in which the subalgebra of exact elements is an MTL-chain. Recently it was proven [7] that IVMTL (and all other interval-valued counterparts of fuzzy logics that satisfy the real-chain embedding property [1,4]) is even standard complete: we can further restrict the semantics to triangle algebras on $\mathcal{L}^I$, which is a lattice on the set of subintervals of the unit interval.

Remark that it is of course also possible to extend IVML with the axiom scheme $(\phi \to \psi) \vee (\psi \to \phi)$. The resulting logic is sound and complete with respect to prelinear triangle algebras. It was proven in [6] that prelinear triangle algebras are exactly triangle algebras in which the subalgebra of exact elements is a Boolean algebra. That is why we called this logic Interval-Valued Classical Propositional Calculus (IVCPC). Using the pseudo-chain completeness in this case shows that IVCPC is actually three-valued (because a (non-trivial) linear Boolean algebra has two elements).

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